Low Emittance Electron Injectors

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1.1 Why low emittance electron injectors needed?

**XFEL**

Seed Laser → Modulative Section → Radiative Section → HGHG FEL

**Dispersive Section** → Electron Beam Input → Electron Beam Output

**Gain Length**

$$L_{g,\text{min}} \approx 20 \frac{\epsilon_{n,x}^{5/6} [\mu m] g^{1/2} [cm]}{I_p^{1/2} [kA] \lambda_x^{2/3} [\AA]}$$

**Wavelength**

$$\lambda_{x,\text{min}} \approx 3 \times 10^3 \frac{\epsilon_{n,x}^{5/4} [\mu m] g^{3/4} [cm]}{I_p^{3/4} [kA] L_w^{3/2} [m]}$$

**Requirement**

$$\epsilon < \frac{\lambda_s}{4\pi} \quad \epsilon_n = \beta \gamma \epsilon$$

**Luminosity**

$$L \propto \eta_{RF} \frac{P_{RF}}{E_{cm}^{3/2}} \sqrt{\frac{\delta_{BS,y}}{\epsilon_{n,y}}} \sqrt{\frac{\sigma_z}{\beta_y}} \propto \eta_{RF} \frac{P_{RF}}{E_{cm}} \sqrt{\frac{\delta_{BS}}{\epsilon_{n,y}}} \sqrt{\frac{\sigma_z}{\beta_y}}$$

**Thomson or Compton Scattering**

**Brightness**

$$B_x = \frac{3n_x \cdot 0.1\% \cdot \gamma^2}{2 \cdot 2.36 \sigma_T / 100 fs \cdot \epsilon_{n,x} \epsilon_{n,y} / (1 mm \cdot mrad)^2}$$
1.2 Some Basic Concepts of Emittance

- **Concept:**
  - Emittance is used to describe the super-volume in the 6-dimensional phase space, as coordinates $(q_1, q_2, q_3)$ and momenta $(p_1, p_2, p_3)$ in Cartesian coordinate systems.
  - If the particles inside any closed surface $s(t)$ in the phase space, surfer only conservative forces, the volume $v(t)$ enclosed by the surface $s(t)$ will be invariant with time.
  - Emittance is the volume $v(t)$ occupied particles of a bunch.

- **Without coupling between $q_1, q_2, q_3$:**
  - Phase space: $(q_1, p_1), (q_2, p_2), (q_3, p_3)$
  - Emittance is a area of the phase space.
RMS Emittance

RMS
\[ \langle x^2 \rangle = \sum_{n} \frac{x^2}{n} - \left( \sum_{n} \frac{x}{n} \right)^2, \quad \langle p_x^2 \rangle = \sum_{n} \frac{p_x^2}{n} - \left( \sum_{n} \frac{p_x}{n} \right)^2. \]

RMS Normal Emittance
\[ \langle x p_x \rangle = \frac{\sum x p_x}{n} - \frac{\sum x \sum p_x}{n^2}, \]
\[ \varepsilon_n = \pi - m_e c \cdot \sqrt{\langle x^2 \rangle \langle p_x^2 \rangle - \langle x \cdot p_x \rangle^2} \]

Here \( m_e c \) means that \( p_x = \beta_x \gamma \). In most cases, \( \varepsilon_n \) is expressed in \( \pi \cdot m_e c \cdot \mu m \).

RMS Geometrical Emittance
\[ \varepsilon = \pi \cdot \sqrt{\langle x^2 \rangle \langle x'^2 \rangle - \langle x \cdot x' \rangle^2} \]
\( x' = p_x / p_z = \beta_x / \beta_z \) is the divergence angle, and \( \varepsilon \) is expressed in \( \pi \cdot \text{mm} \cdot \text{mrad} \).

Other expressions of RMS Emittance
\[ \varepsilon_n = 4 \cdot \pi \cdot m_e c \cdot \sqrt{\langle x^2 \rangle \langle p_x^2 \rangle - \langle x \cdot p_x \rangle^2} \]
\[ \varepsilon = 4 \cdot \pi \cdot \sqrt{\langle x^2 \rangle \langle x'^2 \rangle - \langle x \cdot x' \rangle^2} \]

Assume all the particles have same \( P_z \)
\[ \varepsilon_n = \beta \gamma \cdot \varepsilon \]

What's difference between them?
RMS Emittance

A gaussian phase-space distribution may be specified as:

$$\Psi(x,p_x) = \frac{m_e c}{2 \varepsilon_n} \exp \left( -\frac{\pi^2 m_e^2 c^2 (x^2) \langle p_x^2 \rangle}{4 \varepsilon_n^2} \left\{ \frac{x^2}{\langle x^2 \rangle} - 2 \frac{\langle x p_x \rangle}{\langle x^2 \rangle \langle p_x^2 \rangle} x p_x + \frac{p_x^2}{\langle p_x^2 \rangle} \right\} \right)$$

And

$$\int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dp_x \Psi(x,p_x) = 1$$

To simplify the analysis, $$\langle x p_x \rangle = 0$$

normal emittance is

$$\psi(x,p_x) = \frac{1}{2 \pi \sqrt{\langle x^2 \rangle \langle p_x^2 \rangle}} \exp \left[ -\frac{1}{2} \left( \frac{x^2}{\langle x^2 \rangle} + \frac{p_x^2}{\langle p_x^2 \rangle} \right) \right]$$

And Consider an ellipse in $$(x, p_x)$$ space defined by:

$$\frac{x^2}{\langle x^2 \rangle} + \frac{p_x^2}{\langle p_x^2 \rangle} = K^2$$

The area of this ellipse

$$A(K) = \pi \cdot m_e c \cdot K^2 \cdot \sqrt{\langle x^2 \rangle \langle p_x^2 \rangle}$$

The fraction of the bunch within this ellipse is readily computed:

$$F(K) = \int_0^K k \, dk \exp \left( -\frac{k^2}{2} \right) = 1 - \exp \left( -\frac{K^2}{2} \right)$$

*For K=1, the ellipse has an area equal to the normalized RMS emittance, and contains 39.35% of the particles. The maximum x coordinate of the ellipse is the RMS value of x, while the maximum p_x coordinate of the ellipse is the RMS value of p_x.

*For K=2, the ellipse has an area equal to four times the normalized RMS emittance, and contains 86.47% of the particles.

RMS Emittance and Electron Beam

The transverse momentum $p_x$ & the divergence angle $x'$

$$p_x = m v_x = m \frac{dx}{dt} = m \frac{dx}{dz} \frac{dz}{dt} = m v_z \cdot x' = p_z \cdot x'$$

The RMS geometrical emittance will decrease with the particles being accelerated, but the RMS normal emittance will remain the same.

$$x' = \frac{dx}{dz} = \tan(\theta) \approx \theta$$

The RMS geometrical emittance will decrease with the particles being accelerated, but the RMS normal emittance will remain the same.
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• Photo-electric emission:

1) photo-current is proportional to photon number (if photon energy not changed)

2) The maximum kinetic energy of the photo-electron is proportional to photon
\[ <E_{\text{kin}},/ >_{\text{flat}} = \frac{h \nu - \phi}{3} \]

Metal: Mg 0.4%, Cu 0.05%

Semiconductor:
alkali-based Na2KSb:Cs, K2CsSb,
Cs2Te 266nm 6%~12%, 251nm 16%
GaAs 2.55eV (486nm) 14%
GaAs:Cs 2.3eV 0.26%
Secondary emission:

When the primary electron hits the cathode, atoms will be shocked strongly. Some electrons can cross the potential well and become secondary electrons.

\[
\delta_i(E_0) = \frac{-B}{\varepsilon} \int_0^{\min(R(E_0),d)} \frac{dE(x)}{dx} e^{-ax} dx
\]
1.4 Electron Guns

DC Gun:

- Triode structure: Cathode, anode, and grid electrode to control the emission.
- Wehnelt electrode for the collimated space charge flux.
- The pulse length is limited down to ~1ns due to the switching speed of the driver circuit. The bunch length is more than 1ns.
- This long pulse can be considered to be a continuous beam, in which the gun is operated in the space charge limit.
From Timothy J. Waldron’s Presentation, Functional Requirements for IMRT
Figure 1.4: SEM pictures of a FEA and a cross-cut view into a single emitter coil. The lateral size of the tip is about one micron, the pitch between tips a few microns. (Images courtesy of PSI LEG)
Multipactor Gun

Electron Energy (keV)

SE Yield of Diamond

SEEP Capsule (Ben-Zvi 06)
Thermionic RF Gun

- High brightness \( B_n = \frac{2I_p}{\varepsilon_{nx}\varepsilon_{ny}} \)
- Simple
- Alpha magnet needed to compress the bunch length and serve as a momentum filter
- Electron back bombardment effect
  - Pulse length: cathode damage and beam loading
  - Repeat rate: cathode damage
  - Energy spread: beam loading
  - Emittance: low electric field at cathode

From Chuanxiang tang’s PhD thesis, 1996
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Emittance in Photoinjectors

\[ \varepsilon_n \sim \sqrt{\varepsilon_{th}^2 + \varepsilon_{sc}^2 + \varepsilon_{rf}^2 + \varepsilon_{mp}^2 + \varepsilon_{Bz}^2 + 2\eta\varepsilon_{sc}\varepsilon_{rf}} \]

\( \varepsilon_{th} \)     Initial Emittance or Thermal Emittance
\( \varepsilon_{sc} \)     Space Charge Emittance
\( \varepsilon_{rf} \)     RF Emittance
\( \varepsilon_{mp} \)     Multi-pole mode of RF field caused Emittance
\( \varepsilon_{Bz} \)     Emittance because of Magnetic field Bz at cathode surface
\( \eta \)     Coupling between space charge effect and RF effect

Emittance Consideration in a Photoinjector

- RF cavity optimization
- Electric field gradient
- Cathode: material and its surface
- Bunch charge and its shape
- Laser pulse (space and time)
- Coil design (emittance compensation)
- Time jitter (time dependence electric field)
Energy to Laser Phase

![Graph showing the relationship between laser phase and output energy for different electric field strengths.](image)
Emittance to Solenoid Strength

\[ R_{\text{perp}}(B, z) \]

\[ R_{\text{par}}(B, z) = \frac{R_{\text{par}}(B, z)}{R_{\text{perp}}(B, z)} \]

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\[ R_{\text{par}}(B, z) = \frac{R_{\text{par}}(B, z)}{R_{\text{par}}(B, z)} \]
Emittance vs Bz and Laser radius

Emittance vs Bz & Laser radius when laser phase is 20Deg

Strength of emittance compensation solenoid / Gauss

Laser radius / cm
Emittance vs Bz and laser injection phase

Emittance vs Bz & Laser phase when laser radius is 1.4mm

Strength of emittance compensation solenoid /Gauss

Laser injection phase /Degree
New RF Design of the LCLS Gun

- 15 MHz mode separation
- Dual Feed
- Racetrack shape
- The cell-to-cell iris: the radius of the iris increased and the disk thickness reduced. The profile of the disk iris was modified from circular to elliptical to reduce the surface field.
- Z-coupling of RF feeds
- Laser port holes
- Probe holes
## 2.3 Electric Field and Emittance

### Table:

<table>
<thead>
<tr>
<th>Field Strength (MV/m)</th>
<th>Rise Time (ps)</th>
<th>Phase (Degree)</th>
<th>Radius (mm)</th>
<th>Magnetic Field (Gauss)</th>
<th>Length (cm)</th>
<th>Longitudinal Emittance (μm)</th>
<th>Transverse Emittance (μm)</th>
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<tr>
<td>300</td>
<td>0.7</td>
<td>44</td>
<td>0.57</td>
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<tr>
<td>300</td>
<td>0</td>
<td>46</td>
<td>0.57</td>
<td>5422</td>
<td>257</td>
<td>0.43</td>
<td>0.55</td>
</tr>
<tr>
<td>250</td>
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<td>44</td>
<td>0.62</td>
<td>4602</td>
<td>246</td>
<td>0.45</td>
<td>0.65</td>
</tr>
<tr>
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<td>0</td>
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<tr>
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<td>0.75</td>
<td>3780</td>
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<td>0.51</td>
<td>0.72</td>
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<tr>
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<td>2710</td>
<td>141</td>
<td>0.50</td>
<td>0.96</td>
</tr>
</tbody>
</table>
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2.4 Thermal Emittance of Metal Photocathode

The normal Emittance

\[ \varepsilon_{n,\text{rms}} = \frac{1}{m_0 c} \sqrt{\langle x^2 \rangle \langle p_x^2 \rangle - \langle xp_x \rangle^2} \]

For flat photocathode surface

\[ \varepsilon_{\text{thermal, rms}} = x_{\text{rms}} \frac{p_{\text{rms,x}}}{m_0 c} = x_{\text{rms}} \frac{\sqrt{2m_0 \langle E_{\text{kin,x}} \rangle}}{m_0 c} \]

*By He Xiaozhong, Tang Chuanxiang, et al., High Energy Physics and Nuclear Physics Vol. 28, No. 9, 2004 9*
The statistic average kinetic energy of electron from surface

\[ \langle E_{\text{kin},x} \rangle \approx \frac{1}{6} [h \nu - \Phi + \alpha \sqrt{\beta E(\phi)} ] \]

The thermal emittance of an idea flat cathode:

\[ \varepsilon_{n,\text{rms},\text{flat}} = \frac{R}{2} \sqrt{\frac{h \nu - \phi}{3m_0 c^2}} \]


Compared with measured results

<table>
<thead>
<tr>
<th>测量条件</th>
<th>镁阴极</th>
<th>铜阴极</th>
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<tbody>
<tr>
<td>场强</td>
<td>100MV/m</td>
<td>场强</td>
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<tr>
<td>相位</td>
<td>~15°</td>
<td>相位</td>
</tr>
<tr>
<td>场增强因子</td>
<td>1, 4</td>
<td>场增强因子</td>
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<table>
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<th>测量结果</th>
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<th>铜阴极</th>
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<tr>
<td>Graves 式结果</td>
<td>~0.5, 1.2 mmmrad</td>
<td>~0.5 mmmrad</td>
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<td>Clendenin 式结果</td>
<td>0.78, 0.84 mmmrad</td>
<td>0.41, 0.54 mmmrad</td>
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<td>我们计算的结果</td>
<td>0.55, 0.59 mmmrad</td>
<td>0.29, 0.38 mmmrad</td>
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<tr>
<td>我们计算的结果</td>
<td>0.45, 0.49 mmmrad</td>
<td>0.24, 0.31 mmmrad</td>
</tr>
</tbody>
</table>

*X. J. Wang, et al. Mg cathode and its thermal emittance[J]. LINAC02. 2002
Thermal emittance of metal photo-cathode due to surface roughness

The surface roughness causes the parallel kinetic energy increasing compared with the flat surface. If we assume that the parallel and the vertical kinetic energy of the electron emitted from the rough surface are equal, the thermal emittance will be:

\[ \varepsilon_{n,x} = \frac{R}{2} \cdot \sqrt{\frac{4(h\nu - \phi)}{9m_0c^2}} \]
The emittance caused by the electric field at the rough surface

The stray electric field because of the micro-surface*:

\[
E_x = E_{rf} \sin \theta_{rf} \sum_{n=0}^{\infty} \frac{2n\pi a}{p} c_n e^{-2n\pi z/p} \sin(2n\pi x/p)
\]

\[
E_z = E_{rf} \sin \theta_{rf} \left[ 1 - \sum_{n=0}^{\infty} \frac{2n\pi a}{p} c_n e^{-2n\pi z/p} \cos(2n\pi x/p) \right]
\]

\[
z = a \cos\left(\frac{2\pi x}{p}\right)
\]

\[ \varepsilon_{n,x,\text{rough}} = \frac{R}{2} \cdot \sqrt{\frac{\pi e_0 f(a_u) E_{r,f} \sin \theta_{r,f}}{4m_0 c^2}} \]

\[ f(a_u) = a_u - 0.59a_u^2 + 0.023a_u^3 \]
Copper cathode (left) and Magnesium cathode (right) cut by diamond without polishing
Copper cathode (lower) and Magnesium cathode (upper) cut by diamond without (left) and with (right) polishing.
## Emittance caused by roughness

<table>
<thead>
<tr>
<th></th>
<th>宏观粗糙度</th>
<th>微观粗糙度</th>
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<tbody>
<tr>
<td>周期</td>
<td>8μm</td>
<td>400nm</td>
</tr>
<tr>
<td>幅度</td>
<td>70nm</td>
<td>5nm</td>
</tr>
<tr>
<td>场强</td>
<td>50MV/m</td>
<td>50MV/m</td>
</tr>
<tr>
<td>粗糙度热发射度</td>
<td>~0.27mrrad/mm</td>
<td>~0.09mrrad/mm</td>
</tr>
</tbody>
</table>
Timing Jitter Measurement
2.6 Coil

Bz at different current

Bz(x=0,y=0)

Bz(x=0,y=-4,-2,0,2,4mm)-Bz(x=0,y=0)
Evolution of the two kinds of emittance in a general photoinjector beamline
Correlated and Slice Emittances in Space Charge Dominated Photoinjectors*

- **Slice Emittance**: A transverse cross section of the beam is called a slice, and the emittance of a slice is called slice emittance. Slice emittance consists of two parts, one part is the thermal emittance and the other part is due to nonlinear space charge force.

- **Correlated emittance**: The growth of projected emittance is due in large part to the correlation between the phase space angle and the longitudinal position of slices. Normally, this part of projected emittance is called correlated emittance.

- **Emittance compensation**: Correlated emittance evolves as the beam propagates in the beamline, and can be eliminated at one or more specific points through selecting proper parameters of the beamline. This process of eliminating correlated emittance is called emittance compensation.

*He Xiaozhong, Tang Chuanxiang, et al, to be published at NIM A
Transverse Particle Motion Equation

\[ \frac{d}{dt}(p_r) = e \left( E_r - \beta_z c B_\theta \right) = e E_r \left( 1 - \beta_z^2 \right) \approx \frac{e E_r}{\gamma^2} \]

\[ r''(\zeta, z) = \frac{e^2 \lambda(\zeta)}{2\pi \varepsilon_0 \beta^2 \gamma^3 mc^2} \left( \frac{r_0}{a_0} \right)^2 \frac{1}{r(z)} \]

\[ r_e = \frac{e^2}{4\pi \varepsilon_0 mc^2} \quad k_p^2 = \frac{4\pi r_e n_b(\zeta, z)}{\beta^2 \gamma^3} \]

\[ r''(\zeta, z) = \frac{1}{2} k_{p0}(\zeta) r_0^2 \frac{1}{r(z)} \quad k_{p0}(\zeta) = k_p(\zeta, z = 0) \]
Slice Emittance and The Phase Space

\[ r(\zeta, z) = r_0 + \frac{1}{4} k_{p0}^2 (\zeta) r_0 z^2 \]

\[ r'(\zeta, z) = \frac{1}{2} k_{p0}^2 (\zeta) r_0 z \]

(a) \[ \frac{r}{r_0} \]

(b) \[ \frac{r'}{k_{p0,\text{max}} a_0} \]
After a thin lens at $z=z_l$ with focus of $f$

\[
\begin{align*}
    r(\zeta, z) &= r_0 + \frac{1}{4} k_{p0}^2 (\zeta) r_0 (z + z_l)^2 - \frac{z}{f} \left( r_0 + \frac{1}{4} k_{p0}^2 (\zeta) r_0 z_l^2 \right) \\
    r'(\zeta, z) &= \frac{1}{2} k_{p0}^2 (\zeta) r_0 (z + z_l) - \frac{r_0}{f} \left( 1 + \frac{1}{4} k_{p0}^2 (\zeta) z_l^2 \right)
\end{align*}
\]
space charge dominated waist and ballistic waist (cross-over)

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Conditions of The Emittance Compensation for a Laminar Beam

Emittance compensation means all of the slices are aligned in the phase space.

If a thin lens is put at position \( z_l \), emittance compensation will happen at position \( z \) downstream from

\[
\frac{d}{dk^2} \left( \frac{r'}{r} \right) = 0
\]

\[
\frac{1}{f} = 2 \frac{z + z_l}{z^2}
\]
Envelope Equation

Linear model: space charge force, rf accelerating, external focus and initial emittance

\[
\sigma'' + \frac{\gamma'}{\gamma} \sigma' + \frac{\eta}{8} \left( \frac{\gamma'}{\gamma} \right)^2 \sigma = \frac{\varepsilon_r^2}{\sigma_r^3} + \frac{r_e \lambda(\zeta)}{\gamma^3 \sigma_r}
\]

Envelope evolution

External focus and the rf focus

Accelerating

Initial emittance

Space charge

Envelope is defined as:

\[
\sigma_x = \sqrt{\langle x^2 \rangle}
\]
Without Accelerating and Initial Emittance

A particular solution is got when $\sigma_{r, r''} = 0$:

Perturbation around the Brillouin Flow:

$$\delta \sigma_r''(\zeta, z) + k_\beta \delta \sigma_r(\zeta, z) = -\frac{r_e \lambda(\zeta)}{\gamma^3 \sigma_{eq}^2(\zeta)} \delta \sigma_r(\zeta, z)$$

$$\delta \sigma_{eq}(\zeta) = \frac{1}{k_\beta} \sqrt{\frac{r_e \lambda(\zeta)}{\gamma^3}}$$

$$\delta \sigma_r(\zeta, z) = \sigma_r(\zeta, z) - \sigma_{eq}(\zeta, z)$$

$$\delta \sigma_r''(\zeta, z) + 2k_\beta \delta \sigma_r(\zeta, z) = 0$$
Perturbation around Brillouin Flow

Initial condition: \[ \sigma_r (\zeta, 0) = \sigma_{r0}, \quad \sigma'_r (\zeta, 0) = 0 \]

Solution:
\[ \sigma_r (\zeta, z) = \sigma_{r0} + \left[ \sigma_{r0} - \sigma_{eq} (\zeta) \right] \cos \left( \sqrt{2}k_\beta z \right) \]
\[ \sigma'_r (\zeta, z) = -\sqrt{2}k_\beta \left[ \sigma_{r0} - \sigma_{eq} (\zeta) \right] \sin \left( \sqrt{2}k_\beta z \right) \]

Oscillation frequency: \[ \sqrt{2}k_\beta \]

Projected emittance and the emittance compensation
With Accelerating

A particular solution

\[
\sigma_{\text{inv}}'(\zeta, z) = -\frac{1}{\gamma(z)} \sqrt{\frac{r_e \lambda(\zeta)}{(1+\eta/2)\gamma(z)}}
\]

\[
\gamma \sigma_{\text{inv}}'(\zeta)/\sigma_{\text{inv}}(\zeta) = -\gamma'/2
\]
is independent of \(\zeta\) and \(z\), which means the rates of slope for different slices and at different positions are same.
Perturbation around the Invariant Envelope

\[ \delta \sigma''_r + \left( \frac{\gamma'}{\gamma} \right) \delta \sigma'_r + \frac{1+\eta}{4} \left( \frac{\gamma'}{\gamma} \right)^2 \delta \sigma_r = 0 \]

\[ \sigma_r(z) = \sigma_{inv} + (\sigma_{r0} - \sigma_{inv}) \cos \left[ \frac{\sqrt{1+\eta}}{2} \ln \left( \frac{\gamma_0}{\gamma(z)} \right) \right] \]

\[ \sigma'_r(z) = \frac{\sqrt{1+\eta}}{2} \gamma(z) \left( \sigma_{r0} - \sigma_{inv} \right) \sin \left[ \frac{\sqrt{1+\eta}}{2} \ln \left( \frac{\gamma_0}{\gamma(z)} \right) \right] \]
3.3 The evolution of the emittance

- The correlated projected emittance:

\[ \epsilon_{nr} = \beta_a \gamma_a \sigma_{ra}^2 \sqrt{c_1^2 \Delta \sigma_{nr,0}^2 + c_2^2 \Delta \sigma'_{nr,0}^2 + 2c_1c_2 \Delta \sigma_{nr,0} \Delta \sigma'_{nr,0}}, \]

where \( c_1 = \frac{\partial (\sigma'_{nra}/\sigma_{nra})}{\partial \sigma_{nra,0}} \) and \( c_2 = \frac{\partial (\sigma'_{nra}/\sigma_{nra})}{\partial \sigma'_{nra,0}} \).

- The correlated slice emittance:

\[ \epsilon_{nr} = \beta_u \gamma_u R_u^2 \sqrt{c_3^2 g(\Delta r_{n,0}^2) + c_4^2 g(\Delta r'_{n,0}^2) + 2c_3c_4 g(\Delta r_{n,0} \Delta r'_{n,0})}, \]

where \( c_3 = \frac{\partial (R'_{nu}/R_{nu})}{\partial R_{nu,0}} \), \( c_4 = \frac{\partial (R'_{nu}/R_{nu})}{\partial R'_{nu,0}} \) and \( g(f(\lambda)) = \int_0^1 f(\lambda) \lambda^3 d\lambda \).

Evolution of the two kinds of emittance in a drift tube beamline

The evolution of the two kinds emittance in drift space can be depicted analytically as a function only of the propagating distance and several parameters determined by the initial phase space.

- **Initial parameter of the electron beam**
  - Kinetic Energy (MeV) 4.5
  - Transverse distribution uniform
  - Hard edge Radius(mm) 4.7
  - Divergence at hard edge (mrad) -5.3
  - Longitudinal distribution flat top
  - Pulse length (ps) 20
  - Current (A) 100
  - Initial energy spread 0
  - Initial emittance $\varepsilon_{nr}(z = 0)(\mu m)$ 12.0 (for project emittance)
    4.0 (for slice emittance)
Correlated project emittance when $\Delta \sigma_{nr} \leq 0$, and Correlated slice emittance when $\Delta r_{n} \leq 0$, simulated by PARMELA, compared with the analytical result.
Correlated project emittance when $\Delta \sigma_{nr} = 0$, and Correlated slice emittance when $\Delta r_n = 0$, simulated by PARMELA, compared with the analytical result.
Evolution of the two kinds of emittance in a beamline with solenoid, drift tube and accelerating linac section

- Magnetic length of solenoid (m) 0.22
- Length of the drift tube (m) 0.78
- Peak magnetic field on axis (Gauss) 2200
- Gradient of linac section (MV/m) 17.0
- Length of the linac section (m) 6.0
Correlated project emittance when $\Delta \sigma_{n_r} = 0$, and Correlated slice emittance when $\Delta r_n = 0$, simulated by PARMELA, compared with the analytical result.
Correlated project emittance when $\Delta \sigma_{nr} = 0$, and Correlated slice emittance when $\Delta r_n = 0$, simulated by PARMELA, compared with the analytical result.
3.4 The Emittance Inside a Slice & Wave Breaking

Wave Breaking:

1) $r(z)$ independent of $r(0)dr(z)/dr_0 = 0$

2) Transverse momentum $p_r(r)$ with multi-solution
Correlated slice emittance when $\Delta r_n = 0$, simulated by PARMELA for initial emittance of 4.2 mm.mrad and 12 mm.mrad, compared with the analytical result.
The phase space evolution for initial emittance of 4.2 mm.mrad (lower) and 12 mm.mrad (upper)
Accelerator Lab of Tsinghua University
Thanks!